

LOCAL CONVEXITY PROPERTIES OF THE TRIANGULAR RATIO METRIC BALLS

PARISA HARIRI, RIKU KLÉN, AND MATTI VUORINEN

ABSTRACT. We study local convexity properties of the triangular ratio metric balls in proper subdomains of the real coordinate space. We also study inclusion properties of the visual angle metric balls and related hyperbolic type metric balls in the complement of the origin and the upper half space.

1. INTRODUCTION

The hyperbolic metric has become an important tool in geometric function theory. It works well in simply connected subdomains of the complex plane, because we can use the Riemann mapping theorem to map such domains onto the unit disk, where explicit formulas are known [B1, KL]. In higher dimensions ($n \geq 3$) no counterpart exists and thus there is need for other methods. One approach is to generalize the hyperbolic metric for higher dimensions in such a manner that the generalized metric is comparable with the hyperbolic metric when the domain is the upper half plane or the unit ball

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}, \quad \mathbb{B}^n = \{z \in \mathbb{R}^n : |z| < 1\}.$$

We call these generalizations hyperbolic type metrics. One of the first hyperbolic type metrics, the quasihyperbolic metric, was introduced by Gehring and Palka in the 1970's [GP]. Gradually the quasihyperbolic metric found numerous applications, see e.g. [GH]. During the past two decades many authors have introduced various other hyperbolic type metrics, [HIMPS], [KLVW], [B2], [S], [HMM].

However, it is not clear whether one of these hyperbolic type metrics is better than the other one or which hyperbolic type metric should be applied for a specific application. The natural line of research in this situation is to compare the geometries defined by two hyperbolic type metrics to each other. In this article we study so called triangular ratio metric or s -metric, which is defined as follows for a domain $G \subset \mathbb{R}^n$ and $x, y \in G$:

$$(1.1) \quad s_G(x, y) = \sup_{z \in \partial G} \frac{|x - y|}{|x - z| + |z - y|} \in [0, 1].$$

This metric has been studied in [CHKV, HKLV]. For a metric space (G, m) we define the metric ball for $x \in G$ and $r > 0$ by $B_m(x, r) = \{y \in G : m(x, y) < r\}$. We study the metric balls defined by the triangular ratio

File: hkv20151118.tex, printed: 2015-11-19, 2.17

2010 *Mathematics Subject Classification.* 51M10(52A20).

Key words and phrases. triangular ratio metric ball, visual angle metric, local convexity.

metric. The behaviour of a metric can be studied in many different ways. Our goal is to examine the geometric properties of the metric space (G, s_G) by discussing local convexity properties of the metric balls $B_{s_G}(x, r)$. We find sharp bounds r_0 for the radius of the ball such that $B_{s_G}(x, r)$, $r \in (0, r_0)$, is convex or starlike with respect to x . Similar local convexity results for other hyperbolic type metrics can be found in [HKLv, K1, K2, K3, K4, KRT]. We also compare different hyperbolic type metrics by considering the inclusion of metric balls. For some other hyperbolic type metrics similar research is carried out in [KV1, KV2].

Our main results are the following:

Theorem 1.2. *Let $G \subsetneq \mathbb{R}^n$ be a domain. If G is starlike with respect to $x \in G$. Then for all $r \in (0, 1)$, $B_s(x, r)$ is starlike with respect to x .*

In addition to the s -metric, we also study other metrics defined on subdomains of \mathbb{R}^n , for example the quasihyperbolic metric k_G , the distance ratio metric j_G , the visual angle metric v_G , and the point pair function p_G which are defined in Section 4. For these metrics we summarize our results in the following theorems.

Theorem 1.3. *Let $x \in \mathbb{R}^n \setminus \{0\}$, $r \in (0, \pi]$, and $m \in \{j, k, |\cdot|, p, q, s\}$. Then we can find the best possible radius $t = t(r)$ such that $B_m(x, t) \subset B_v(x, r)$.*

Theorem 1.4. *Let $x \in \mathbb{H}^n$ and $r \in (0, \pi/2]$. Then*

$$\begin{aligned} B^n(x + (\sec^2 r - 1)x_n e_n, (\tan r)x_n) &\subset B_v(x, r) \\ &\subset B^n\left(x + (2x_n \tan^2 r)e_n, 2x_n \frac{\tan r}{\cos r}\right), \end{aligned}$$

and the Euclidean balls are the best possible. Moreover

$$B^n(x, x_n \sin r) \subset B_v(x, r) \subset B^n\left(x, 2x_n \left(\frac{\tan r}{\cos r} + \tan^2 r\right)\right).$$

This paper may be considered to be a continuation of the earlier studies [HKLv, K1, K2, K3, K4, KMS]. Our main results and their proofs suggest that similar results might be valid for other metrics as well and this offers ideas for further studies of the same topic, for instance for the Apollonian or the Seittenranta metrics [B2, S].

2. STARLIKENESS AND CONVEXITY OF TRIANGULAR RATIO METRIC BALLS

In this section we consider local convexity properties of s -metric balls. We start with $\mathbb{R}^n \setminus \{0\}$ and generalize the results to proper subdomains of \mathbb{R}^n . Before studying local convexity properties we introduce preliminary results.

Given two points x and y in \mathbb{R}^n , the line segment between them is denoted by

$$[x, y] = \{(1 - t)x + ty : 0 \leq t \leq 1\}.$$

The notation $\angle(x, z, y)$ stands for the angle in the range $[0, \pi]$ between the line segments $[x, z]$ and $[y, z]$.

Definition 2.1. Let $G \subsetneq \mathbb{R}^n$ be a domain and $x \in G$. We say that G is starlike with respect to x if for every $y \in G$, $[x, y] \subset G$. The domain G is strictly starlike with respect to x if G is bounded and each ray from x meets ∂G at exactly one point.

Definition 2.2. For a domain $G \subsetneq \mathbb{R}^n$ and a point $x \in G$ we define the maximal starlike domain with respect to x as $G' = \{y \in G : [x, y] \subset G\}$.

Proposition 2.3. Let $G \subsetneq \mathbb{R}^n$ be a domain, $x \in G$ be a point and G' be the maximal starlike domain with respect to x . Then for all $y \in G'$

$$s_G(x, y) = s_{G'}(x, y).$$

In particular, for all $t > 0$

$$B_{s_G}(x, t) = B_{s_{G'}}(x, t).$$

Proof. Fix $x \in G$. By domain monotonicity property of s -metric, since $G' \subset G$, we have $s_{G'}(x, y) \geq s_G(x, y)$. Now it is enough to show that $s_{G'}(x, y) \leq s_G(x, y)$. To see this we first fix $y \in G'$ and assume $z \in \partial G'$ such that

$$s_{G'}(x, y) = \frac{|x - y|}{|x - z| + |z - y|}.$$

If $z \in \partial G$. Then

$$s_G(x, y) \geq \frac{|x - y|}{|x - z| + |z - y|} = s_{G'}(x, y).$$

Therefore we may assume that $z \in \partial G' \setminus \partial G$. It follows that z is in the line through x and tangent to the boundary of G at some point $z_1 \in \partial G$. Now by easy geometric observation we see that $|x - z| + |z - y| \geq |x - z_1| + |z_1 - y|$, and

$$s_G(x, y) \geq \frac{|x - y|}{|x - z_1| + |z_1 - y|} \geq \frac{|x - y|}{|x - z| + |z - y|} = s_{G'}(x, y).$$

□

Proposition 2.4. Let $x, y \in \mathbb{R}^n$, $r > 1$, and assume $y' \in [x, y]$. Then $E(x, y', |x - y'|r)$ is contained in $E(x, y, |x - y|r)$.

Proof. By symmetry of the ellipsoid we may suppose that $n = 2$ and $x = 0$, $y = 1$. Now for $y' \in (0, 1)$, we denote $E_1 = E(0, y, r)$ and $E_2 = E(0, y', r|y'|)$. To verify inclusion $E_2 \subseteq E_1$ we need to show that $|v| \leq |u|$ for $u \in \partial E_1 \cap \{(z, 0) \in \mathbb{R}^2 : z < 0\}$, and $v \in \partial E_2 \cap \{(z, 0) \in \mathbb{R}^2 : z < 0\}$. We obtain

$$|u| + |u - 1| = r \Rightarrow |u| = \frac{r - 1}{2},$$

$$|v| + |v - y'| = |y'|r \Rightarrow |v| = |y'| \frac{r - 1}{2} < |u|. \quad \square$$

Theorem 2.5. Let $x \in \mathbb{R}^n \setminus \{0\}$. For all $r > 0$, $B_s(x, r)$ is starlike with respect to x .

Proof. Assume that $x = ce_1$ with $c > 0$ and $y \in G$ is on the ray from x to the sphere $\partial B_s(x, r)$. Let $|x - y| = t$ then by the law of cosines

$$s_G(x, y) = \frac{|x - y|}{|x| + |y|} = \frac{t}{|x| + |y|} = \frac{t}{|x| + \sqrt{|x|^2 + t^2 - 2|x|t \cos \gamma}},$$

where γ is the angle between two line segments $[0, x]$ and $[x, y]$ and γ is assumed to be fixed. It suffices to show that $s_G(x, y)$ is increasing as function of t . If we define

$$f(t) = \frac{t}{|x| + \sqrt{|x|^2 + t^2 - 2|x|t \cos \gamma}}$$

then

$$f'(t) = \frac{|x|(-t \cos \gamma + |x| + \sqrt{|x|^2 + t^2 - 2|x|t \cos \gamma})}{\sqrt{|x|^2 + t^2 - 2|x|t \cos \gamma}(x + \sqrt{|x|^2 + t^2 - 2|x|t \cos \gamma})^2}$$

which is positive whenever

$$g(t) = -t \cos \gamma + |x| + \sqrt{|x|^2 + t^2 - 2|x|t \cos \gamma}$$

is positive and

$$g(t) \geq -t + |x| + \sqrt{|x|^2 + t^2 - 2|x|t} = -t + |x| + ||x| - t| \geq 0.$$

Hence $s_G(x, y)$ is increasing and $B_s(x, r)$ is starlike with respect to x . \square

Proof of Theorem 1.2. Take $y \in \partial B_s(x, r)$. Assume on the contrary that, $B_s(x, r)$ is not starlike, so there exists $y' \in (x, y)$ such that $s_G(x, y') = r$. Because G is starlike it follows that $y' \in G$. By the definition of s -metric,

$$s_G(x, y) = r = \frac{|x - y|}{|x - z| + |z - y|}, \quad z \in \partial G.$$

Then $\{z : |x - z| + |z - y| = \frac{|x - y|}{r}\}$ gives the equation of the maximal ellipse with foci x, y , similarly $\{z' : |x - z'| + |z' - y'| = \frac{|x - y'|}{r}\}$ defines the ellipse with foci x, y' . By Proposition 2.4 the second ellipse is contained in the first ellipse so the point z' is inside the first ellipse, and it is a contradiction to the starlikeness of the domain. So $B_s(x, r)$ is starlike. \square

Corollary 2.6. *Let $G \subsetneq \mathbb{R}^n$ be a domain. For all $x \in G$ and $r \in (0, 1)$ $B_s(x, r)$ is starlike with respect to x .*

Proof. Let us denote by G' the maximal subdomain of G , which is starlike with respect to x . By the definition of the s -metric the metric balls with center x are equal in the domains G' and G . The assertion follows from Theorem 1.2. \square

Next we continue the study of [HKL] by considering the convexity of triangular ratio metric balls in a general subdomain of \mathbb{R}^n .

Lemma 2.7. [HKL, 3.6, 3.8] *Let $x \in G = \mathbb{R}^n \setminus \{0\}$ and $r \in (0, 1)$. Then $B_s(x, r)$ is (strictly) convex if and only if $r \leq 1/2$ ($r < 1/2$).*

Theorem 2.8. *Let $G \subsetneq \mathbb{R}^n$ be a domain, $x \in G$ and $r \in (0, 1)$. Then $B_s(x, r)$ is convex if $r \leq 1/2$.*

Proof. By [HKL^V, (2.2)] the ball $B_s(x, r)$ is the intersection of balls $B_{s_z}(x, r)$, where s_z is the triangular ratio metric in $\mathbb{R}^n \setminus \{z\}$, $z \in \partial G$, and by Lemma 2.7 each of these balls $B_{s_z}(x, r)$ is convex. The assertion follows as intersection of convex domains is a convex domain. \square

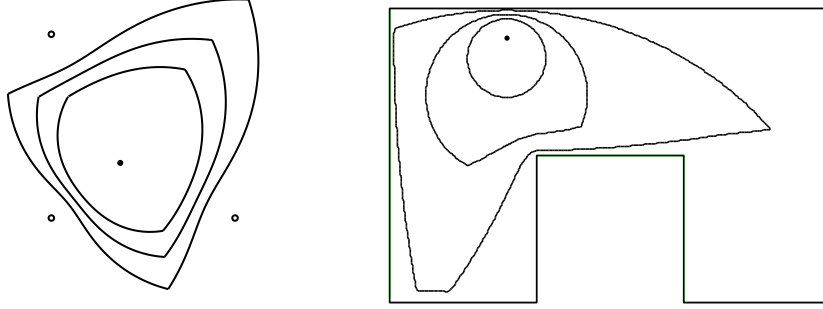


FIGURE 1. Left: The s -metric disks $s(0.75e_1 + 0.6e_2, r)$ in $\mathbb{R}^2 \setminus \{0, 2e_1, 2e_2\}$ with $r = 0.4, r = 0.5, r = 0.6$. Right: s -metric disks in a polygonal domain.

Finally, we make the following conjecture for the balls of the point pair function, which is defined below in 4.4.

Conjecture 2.9. *Let $x \in \mathbb{R}^n \setminus \{0\}$ and $r \in (0, 1)$. Then $B_p(x, r)$ is (strictly) convex if and only if $r \leq \sqrt{2} - 1$.*

3. FORMULA FOR VISUAL ANGLE METRIC IN \mathbb{H}^n

For a domain $G \subsetneq \mathbb{R}^n$, $n \geq 2$, and $x, y \in G$ let

$$(3.1) \quad v_G(x, y) = \sup\{\angle(x, z, y) : z \in \partial G\}.$$

If ∂G is not a proper subset of a line, then v_G defines a metric on G , as shown in [KLVW, Lemma 2.8].

The supremum in (3.1) can be found by a geometric construction if $G = \mathbb{B}^2$. Indeed by [KLVW, Theorem 1.2], by considering the two points z_1 and z_2 of intersection of the ellipses with foci at $0, x$ and $0, y$, respectively, both with focal sum equal to 1, the formula for $v_{\mathbb{B}^2}(x, y)$ is just

$$v_{\mathbb{B}^2}(x, y) = \max\{\angle(x, z_1/|z_1|, y), \angle(x, z_2/|z_2|, y)\}.$$

Here we find an analogue of this formula for \mathbb{H}^2 by finding the points of intersection of two parabolas with foci at x and y , respectively, and both with the real axis $\partial\mathbb{H}^2$ as the directrix, see Figure 2.

For this purpose it is convenient to use horocycles. For two distinct points $x, y \in G$ where $G = \mathbb{B}^2$ or $G = \mathbb{H}^2$, a horocycle through x, y is a Euclidean circle or line through x and y tangent to ∂G .

We consider the problem of finding the center points of two horocycles. These centers are the points of intersection of the parabolas with foci at x and y and directrix $\partial\mathbb{H}^2$. Therefore the formula for centers $z = (z_1, z_2)$ of these horocycles are given by

$$|x - z| = z_2 = |y - z|,$$

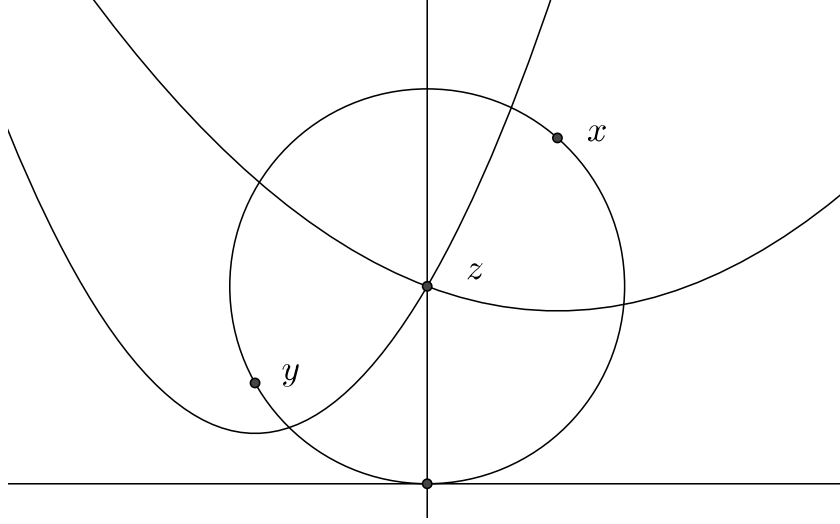


FIGURE 2. The point $z = (z_1, z_2)$ is the intersection of the parabola with focus x and directrix $\partial\mathbb{H}^2$ and the parabola with focus y and directrix $\partial\mathbb{H}^2$. The extremal point $(z_1, 0)$ in the definition of $v_{\mathbb{H}^2}(x, y)$ can be found as the projection of z to $\partial\mathbb{H}^2$.

we have

$$\begin{cases} (x_1 - z_1)^2 + (x_2 - z_2)^2 = z_2^2, \\ (y_1 - z_1)^2 + (y_2 - z_2)^2 = z_2^2. \end{cases}$$

By solving this system of quadratic equations we get

$$z_1 = \frac{x_2 y_1 - x_1 y_2 \pm \sqrt{x_2 y_2} |x - y|}{x_2 - y_2}, \quad x_2 \neq y_2.$$

If $x_2 = y_2$, then $z_1 = (x_1 + y_1)/2$. In terms of this solution in the case $x_2 \neq y_2$, the possible extremal points for the visual angle metric are the two possible points of the form $(z_1, 0) \in \partial\mathbb{H}^2$, and here we choose either $+$ or $-$ whichever corresponds to the smaller imaginary part.

Now by Figure 3

$$v_{\mathbb{H}^2}(x, y) = \pi - \alpha - \beta,$$

$$\alpha = \arctan\left(\frac{x_2}{z_1 - x_1}\right), \quad \beta = \arctan\left(\frac{y_2}{y_1 - z_1}\right).$$

Therefore

(3.2)

$$v_{\mathbb{H}^2}(x, y) \equiv \pi - \arctan\left(\frac{2\sqrt{x_2 y_2} |x - y| \pm (x_2 + y_2)(x_1 - y_1)}{(x_1 - y_1)^2 - 4x_2 y_2}\right) \pmod{\pi}.$$

Another formula for $v_{\mathbb{H}^2}(x, y)$ can be derived from

$$z_2 = \frac{|x - y|}{2(x_2 - y_2)^2} (|x - y|(x_2 + y_2) \mp 2(x_1 - y_1)\sqrt{x_2 y_2})$$

and the law of cosines together with the inscribed angle theorem.

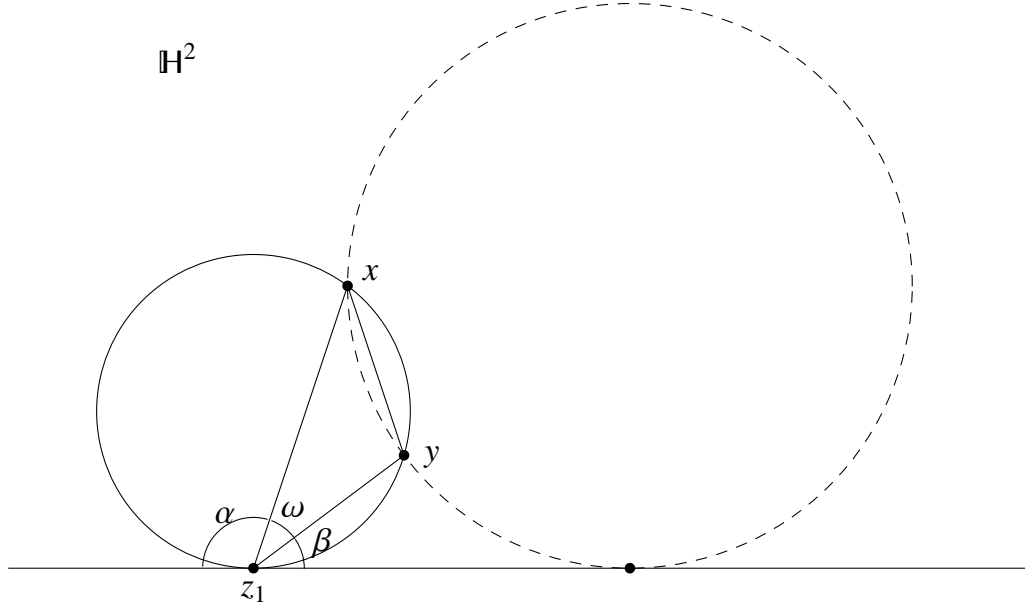


FIGURE 3. Two horocycles through x and y and the extremal point $(z_1, 0)$ with $v_{\mathbb{H}^2}(x, y) = \angle(x, z_1, y)$.

4. INCLUSION PROPERTIES OF METRIC BALLS IN $\mathbb{R}^n \setminus \{0\}$ AND \mathbb{H}^n

In this section we study inclusions of the visual angle metric balls and other metric balls. We begin by defining the metrics we use.

4.1. Quasihyperbolic metric. Let G be a proper subdomain of \mathbb{R}^n . For all $x, y \in G$, the quasihyperbolic metric k_G is defined as

$$k_G(x, y) = \inf_{\gamma} \int_{\gamma} \frac{1}{d(z, \partial G)} |dz|,$$

where the infimum is taken over all rectifiable arcs γ joining x to y in G [GP]. If we assume $x, y \in G = \mathbb{R}^n \setminus \{0\}$ and the angle φ between the line segments $[0, x]$ and $[0, y]$ satisfies $0 < \varphi < \pi$ then by [Vu2, 3.11]

$$(4.2) \quad k_G(x, y) = \sqrt{\varphi^2 + \log^2 \frac{|x|}{|y|}}, \quad G = \mathbb{R}^n \setminus \{0\}.$$

4.3. Distance ratio metric. For a proper open subset $G \subset \mathbb{R}^n$ and for all $x, y \in G$, the distance ratio metric j_G is defined as

$$j_G(x, y) = \log \left(1 + \frac{|x - y|}{\min\{d(x, \partial G), d(y, \partial G)\}} \right).$$

This metric was introduced by F.W. Gehring and B.P. Palka [GP] in a slightly different form and in the above form in [Vu1]. If confusion seems unlikely, then we also write $d(x) = d(x, \partial G)$.

4.4. Point pair function. We define for $x, y \in G \subsetneq \mathbb{R}^n$ the point pair function

$$p_G(x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + 4d(x)d(y)}}.$$

This point pair function was introduced in [CHKV] where it turned out to be a very useful function in the study of the triangular ratio metric. However, there are domains G such that p_G is not a metric: for instance this is the case if $G = \mathbb{B}^2$, [CHKV, Remark 3.1].

4.5. Chordal metric. The chordal metric is defined by

$$\begin{cases} q(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}} & ; x, y \in \mathbb{R}^n, \\ q(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}. \end{cases}$$

Proposition 4.6. *If $G = \mathbb{R}^n \setminus \{0\}$, then the v -balls $B_v(x, r)$, $x \in G$, $r \in (0, \pi)$, are angular domains with vertex at 0.*

Lemma 4.7. *For all $x \in \mathbb{R}^n \setminus \{0\}$, $r \in (0, \pi]$*

$$B_s\left(x, \sin \frac{r}{2}\right) \subset B_v(x, r),$$

and the radius $\sin \frac{r}{2}$ is the best possible.

Proof. By symmetry we may assume that $n = 2$, $x = e_1$. Let $y = te^{ir}$, $t > 0$. Now

$$s_G(x, y) = \frac{|x - y|}{|x| + |y|} = \frac{\sqrt{1 + t^2 - 2t \cos r}}{1 + t} =: f(t).$$

By Proposition 4.6, we want to minimize $f(t)$. Now

$$f'(t) = \frac{(t - 1)(1 + \cos r)}{(1 + t)^2 \sqrt{1 + t^2 - 2t \cos r}},$$

and $f'(t) = 0$ if and only if $t = 1$, so $|y| = |x|$ and $s_G(x, y) = \frac{|x - y|}{2|x|} = \sin \frac{r}{2}$. The sharpness follows from this argument. \square

Lemma 4.8. *For all $x \in \mathbb{R}^n \setminus \{0\}$ and $r \in (0, \pi]$*

$$B_j\left(x, \log\left(1 + 2 \sin \frac{r}{2}\right)\right) \subset B_v(x, r),$$

and the radius $\log(1 + 2 \sin \frac{r}{2})$ is the best possible.

Proof. In the same way as in the proof of Lemma 4.7, we may assume again that $n = 2$, $x = e_1$, let $y = te^{ir}$, $t > 0$. Now

$$j_G(x, y) = \log\left(1 + \frac{|x - y|}{\min\{|x|, |y|\}}\right) = \log\left(1 + \frac{\sqrt{1 + t^2 - 2t \cos r}}{\min\{1, t\}}\right).$$

Define

$$f(t) = \begin{cases} \log\left(1 + \frac{\sqrt{1 + t^2 - 2t \cos r}}{t}\right) & , t \leq 1, \\ \log\left(1 + \sqrt{1 + t^2 - 2t \cos r}\right) & , t > 1. \end{cases}$$

Computation yields

$$f'(x) = \begin{cases} \frac{t \cos r - 1}{t(1 - 2t \cos r + t(t + \sqrt{1 + t^2 - 2t \cos r}))} & , t \leq 1, \\ \frac{t - \cos r}{1 + t^2 - 2t \cos r + \sqrt{1 + t^2 - 2t \cos r}} & , t > 1. \end{cases}$$

By Proposition 4.6, the extremal case takes place when $t = 1$, hence $|y| = |x|$ and $j_G(x, y) = \log \left(1 + \frac{|x-y|}{|x|} \right) = \log \left(1 + 2 \sin \frac{r}{2} \right)$. Therefore $R = \log \left(1 + 2 \sin \frac{r}{2} \right)$ and the proof is complete. The sharpness follows from this proof. \square

Lemma 4.9. *For all $x \in \mathbb{R}^n \setminus \{0\}$ and $r \in (0, \pi]$*

$$B_k(x, r) \subset B_v(x, r),$$

and the radius r is the best possible.

Proof. We assume by symmetry that $n = 2$, $x = te_1$, let $y = e_1$, $t > 0$. Now

$$k_G(x, y) = \sqrt{r^2 + \log^2 \frac{|x|}{|y|}} = \sqrt{r^2 + \log^2 t} =: f(t),$$

and

$$f'(t) = \frac{\log t}{t\sqrt{r^2 + \log^2 t}}.$$

By Proposition 4.6, the extremal case happens when $t = 1$, so $|y| = |x|$ and $k_G(x, y) = r$ and the proof is complete. The sharpness follows from the proof. \square

Lemma 4.10. *For all $x \in \mathbb{R}^n \setminus \{0\}$ and $r \in (0, \pi]$*

$$\mathbb{B}^n(x, R) \subset B_v(x, r),$$

$$R = \begin{cases} |x| \sin r & , r \in (0, \frac{\pi}{2}], \\ |x| & , r \in (\frac{\pi}{2}, \pi], \end{cases}$$

and the radius R is the best possible.

Proof. Fix $x \in \mathbb{R}^n \setminus \{0\}$, and $y \in B_v(x, r)$. We consider two cases. If $r \in (\frac{\pi}{2}, \pi]$ then by Proposition 4.6, $|x - y| \leq |x|$. If $r \in (0, \frac{\pi}{2}]$ then by the law of sines and Proposition 4.6, $|x - y| \leq |x| \sin r$. The sharpness follows from the proof. \square

Lemma 4.11. *For all $x \in \mathbb{R}^n \setminus \{0\}$ and $r \in (0, \pi]$*

$$B_q(x, R) \subset B_v(x, r), \quad R = \min \left\{ \frac{2|x| \sin r/2}{1 + |x|^2}, \frac{|x|}{\sqrt{1 + |x|^2}} \right\},$$

and the radius R is the best possible.

Proof. Fix $x \in \mathbb{R}^n \setminus \{0\}$, and $y \in B_v(x, r)$. We consider two cases. If $r \in (\frac{\pi}{2}, \pi]$ then

$$q(x, y) \leq q(x, 0) = \frac{|x|}{\sqrt{1 + |x|^2}}.$$

If $r \in (0, \frac{\pi}{2}]$ we may assume that $x = e_1$, and let $y = te^{ir}$, $t > 0$. Then by the law of sines

$$q(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}} = \frac{\sqrt{1 + t^2 - 2t \cos r}}{\sqrt{2} \sqrt{1 + t^2}} =: f(t),$$

and

$$f'(t) = \frac{(t^2 - 1) \cos r}{\sqrt{2}(1 + t^2)^{3/2} \sqrt{1 + t^2 - 2t \cos r}}.$$

The extremal case takes place when $t = 1$, therefore $|x| = |y|$, and

$$q(x, y) = \frac{|x - y|}{1 + |x|^2} = \frac{2|x| \sin(r/2)}{1 + |x|^2}.$$

The proof is complete because by Proposition 4.6, $B_v(x, y)$ is an angular domain. The sharpness follows from the proof. \square

Lemma 4.12. *For all $x \in \mathbb{R}^n \setminus \{0\}$ and $r \in (0, \pi]$*

$$B_p(x, R) \subset B_v(x, r), \quad R = \frac{\sin(r/2)}{\sqrt{\sin^2(r/2) + 1}},$$

and the radius R is the best possible.

Proof. By symmetry we may assume that $n = 2$, $x = e_1$. Let $y = te^{ir}$, $t > 0$. By the definition

$$p_G(x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + 4|x||y|}} = \sqrt{\frac{1 + t^2 - 2t \cos r}{1 + t^2 - 2t \cos r + 4t}} =: f(t).$$

By Proposition 4.6, we want to minimize $f(t)$. Now

$$f'(t) = \frac{2(t^2 - 1)}{\sqrt{1 + t^2 - 2t \cos r}(1 + t(4 + t) - 2t \cos r)^{3/2}},$$

and $f'(t) = 0$ if and only if $t = 1$, so $|y| = |x|$ and

$$p_G(x, y) = \frac{|x - y|}{\sqrt{|x - y|^2 + 4|x|^2}} = \frac{\sin(r/2)}{\sqrt{\sin^2(r/2) + 1}}.$$

The sharpness follows from the proof. \square

Proof of Theorem 1.3. The proof follows from Lemmas 4.8, 4.9, 4.10, 4.12, 4.11 and 4.7. \square

Theorem 4.13. *For all $x \in \mathbb{H}^n$ and $r \in (0, \pi/2)$*

$$(4.14) \quad \begin{aligned} B_v(x, r) &\subset B^n(x + (2T^2)x_n e_n, 2T\sqrt{T^2 + 1}x_n) \\ &\subset B^n(x, (2T\sqrt{T^2 + 1} + 2T^2)x_n), \end{aligned}$$

where $T = |\tan(\pi - r)|$. Moreover, the smaller Euclidean ball is the smallest possible containing $B_v(x, r)$.

Note that (4.14) is equivalent to

$$\begin{aligned} B_v(x, r) &\subset B^n\left(x + (2x_n \tan^2 r)e_n, 2x_n \frac{\tan r}{\cos r}\right) \\ &\subset B^n\left(x, 2x_n \left(\frac{\tan r}{\cos r} + \tan^2 r\right)\right). \end{aligned}$$

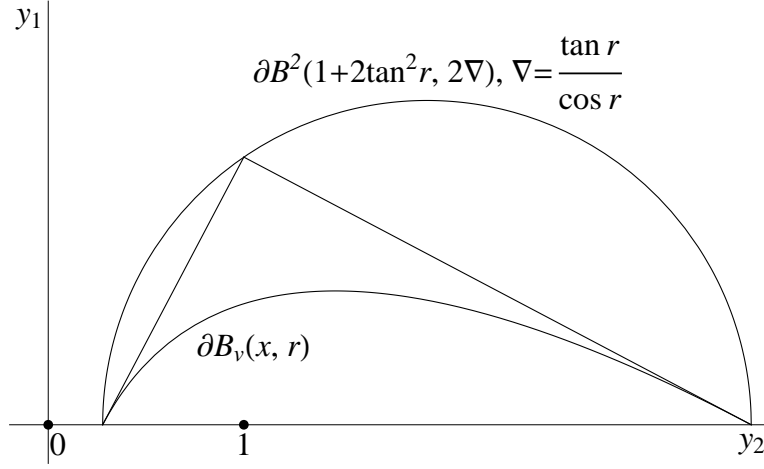


FIGURE 4. Proof of Theorem 4.13.

Proof. It suffices to consider the case $n = 2$. For the first inclusion, let us fix $x = i$. We claim that

$$B_v(i, r) \subset B^2 \left((1 + 2 \tan^2 r) i, 2 \frac{\tan r}{\cos r} \right).$$

By (3.2), we have

$$v_{\mathbb{H}^2}(x, y) \equiv \pi - \arctan \left(\frac{2\sqrt{y_2} \cdot \sqrt{y_1^2 + (y_2 - 1)^2} \mp y_1(1 + y_2)}{y_1^2 - 4y_2} \right) \pmod{\pi}.$$

Writing $v_{\mathbb{H}^2}(i, y) = r$, we conclude that

$$(4.15) \quad y_1 = \begin{cases} \cot r(1 + y_2 - 2\sqrt{y_2} \sec r) =: f_1(y_2, r), \\ \cot r(-1 - y_2 + 2\sqrt{y_2} \sec r) =: f_2(y_2, r). \end{cases}$$

This gives the equation of $\partial B_v(i, r)$, for $y_2 \in [b_1, b_2]$.

Letting $f_1(y_2, r) = f_2(y_2, r) = 0$, we see that

$$(4.16) \quad y_2 = \begin{cases} 2(1 - \sin r) \sec^2 r - 1 =: b_1, \\ 2(1 + \sin r) \sec^2 r - 1 =: b_2. \end{cases}$$

Our next goal is to find the equation of $\partial B^2((1 + 2 \tan^2 r) i, 2 \frac{\tan r}{\cos r})$. Taking $|y - (1 + 2 \tan^2 r) i| = 2 \frac{\tan r}{\cos r}$, gives

$$y_1 = \begin{cases} -\sqrt{4y_2 \sec^2 r - (1 + y_2)^2} =: g_1(y_2, r), \\ \sqrt{4y_2 \sec^2 r - (1 + y_2)^2} =: g_2(y_2, r). \end{cases}$$

By symmetry, it is sufficient to show that $g_2(y_2, r) \geq f_2(y_2, r)$. In order to prove this inequality, it is convenient to estimate the circular arc $g_2(y_2, r)$, by a triangle with vertices b_1, b_2 and $(1, g_2(1, r))$. We can do this estimation because

$$\frac{\partial^2}{\partial y_2^2}(g_2(y_2, r)) = \frac{-4 \sec^2 r \tan^2 r}{(4y_2 \sec^2 r - (1 + y_2)^2)^{3/2}} \leq 0.$$

Denote the above mentioned triangle by $T(y_2, r)$

$$(4.17) \quad T(y_2, r) = \begin{cases} y_1 = \frac{g_2(1, r)}{1-b_1}(y_2 - b_1) =: l_1(y_2, r), & b_1 \leq y_2 < 1, \\ y_1 = \frac{g_2(1, r)}{1-b_2}(y_2 - b_2) =: l_2(y_2, r), & 1 \leq y_2 \leq b_2. \end{cases}$$

We only need to show that $h(y_2, r) = T(y_2, r) - f_2(y_2, r) \geq 0$.

If $b_1 \leq y_2 < 1$, then

$$h(y_2, r) = \frac{1}{\cos r}(y_2 - 1 + (y_2 + 1) \sin r) - \cot r(2\sqrt{y_2} \sec r - 1 - y_2).$$

But $\frac{\partial^2(h(y_2, r))}{\partial y_2^2} = \frac{\csc r}{2y_2^{3/2}} > 0$, for $0 \leq r \leq \pi/2$. Hence $h'(y_2, r)$ is an increasing function of y_2 . Moreover we claim that indeed

$$h'(b_1, r) = \cot r + \sec r + \tan r - \frac{\csc r}{\sqrt{-1 + 2/(1 + \sin r)}} = 0.$$

To see this, it is sufficient to make the following observation

$$\sqrt{\frac{1 - \sin r}{1 + \sin r}} = \frac{\csc r}{\cot r + \sec r + \tan r}.$$

Now it is easy to check that for $0 \leq r \leq \pi/2$, both sides are equivalent to $\csc\left(\frac{\pi}{4} + \frac{r}{2}\right) \sin\left(\frac{\pi}{4} - \frac{r}{2}\right)$. Thus $h'(y_2, r) \geq 0$.

Similarly we can show that

$$h(b_1, r) = 2 \sec r \left(\csc r - 1 - \cot r \sqrt{\frac{1 - \sin r}{1 + \sin r}} \right) = 0.$$

To see this, it suffices to show that

$$\frac{1 - \sin r}{\cos r} = \sqrt{\frac{1 - \sin r}{1 + \sin r}}.$$

It follows easily that for $0 \leq r \leq \pi/2$, both sides are equivalent to

$$\csc\left(\frac{\pi}{4} + \frac{r}{2}\right) \sin\left(\frac{\pi}{4} - \frac{r}{2}\right).$$

Hence $h(y_2, r) \geq 0$. In the same manner for $1 \leq y_2 \leq b_2$,

$$h(y_2, r) = (1 + y_2 + \csc r - y_2 \csc r) \tan r - \cot r(-1 - y_2 + 2 \sec r \sqrt{y_2}).$$

But $\frac{\partial^2(h(y_2, r))}{\partial y_2^2} = \frac{\csc r}{2y_2^{3/2}} > 0$, for $0 \leq r \leq \pi/2$, and

$$h'(b_2, r) = \tan r + \cot r - \sec r - \frac{\csc r}{\sqrt{2 \sec^2 r + 2 \tan r \sec r - 1}} = 0.$$

To see this, it is enough to show that

$$\sqrt{2 \sec^2 r + 2 \tan r \sec r - 1} = \frac{\csc r}{\tan r + \cot r - \sec r},$$

and it is easy to check that for $0 \leq r \leq \pi/2$, both sides are equivalent to $\csc\left(\frac{\pi}{4} - \frac{r}{2}\right) \sin\left(\frac{\pi}{4} + \frac{r}{2}\right)$. Therefore $h'(y_2, r) \geq 0$. We next show that

$$h(b_2, r) = 2 \sec r \left(\csc r - \cot r \sqrt{2 \sec r (\tan r + \sec r) - 1} + 1 \right) = 0.$$

To see this we need to show that

$$\sqrt{2 \sec r (\tan r + \sec r) - 1} = (1 + \csc r) \tan r.$$

In the same manner as in the previous part, we can show that for $0 \leq r \leq \pi/2$, both sides are equivalent to $\csc(\frac{\pi}{4} - \frac{r}{2}) \sin(\frac{\pi}{4} + \frac{r}{2})$, Hence $h(y_2, r) \geq 0$.

An easy computation shows that $g_2(b_1, r) = g_2(b_2, r) = 0$. Hence the Euclidean ball $B^2((1 + 2 \tan^2 r)i, 2 \frac{\tan r}{\cos r})$ is the smallest possible containing $B_v(x, r)$, and this completes the proof for the first inclusion.

For the second inclusion, let

$$y \in \partial B^2 \left(x + (2x_2 \tan^2 r)e_2, 2 \frac{\tan r}{\cos r} x_2 \right).$$

It follows that

$$|y - x_1 e_1 - (1 + 2 \tan^2 r)x_2 e_2| \leq 2 \frac{\tan r}{\cos r} x_2.$$

Therefore

$$\begin{aligned} |y - x| &\leq |y - x_1 e_1 - (1 + 2 \tan^2 r)x_2 e_2| + |-x + x_1 e_1 + (1 + 2 \tan^2 r)x_2 e_2| \\ &\leq 2x_2 \left(\frac{\tan r}{\cos r} + \tan^2 r \right), \end{aligned}$$

and $y \in \partial B^2(x, 2x_2(\frac{\tan r}{\cos r} + \tan^2 r))$. □

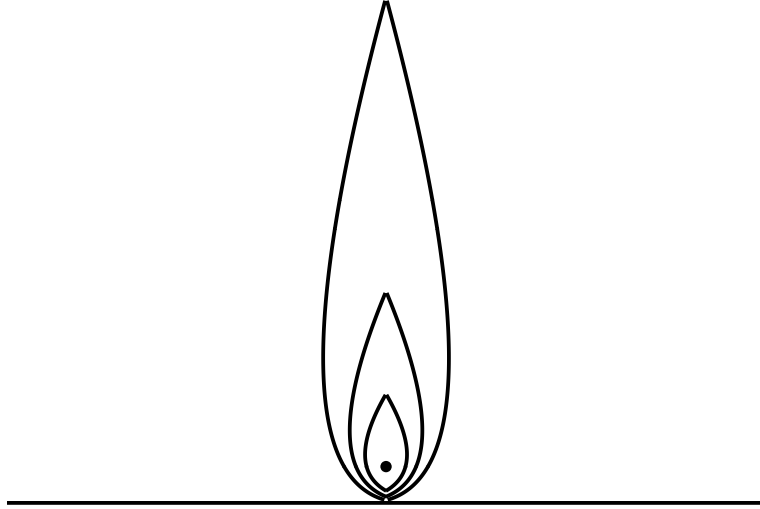


FIGURE 5. $\partial B_v(i, r)$, in the upper half space, for $r = \pi/6, \pi/4, \pi/3$.

Remark 4.18. By the proof of Theorem 4.13 we observe that $\partial B_v(x, r)$ is not smooth for $0 \leq r \leq \pi$. By (4.15),

$$f'_2(b_1^+, r) = \cot r \left(\frac{\sec r}{\sqrt{2 \sec^2 r - 2 \tan r \sec r - 1}} - 1 \right),$$

and

$$f'_1(b_1^+, r) = -\cot r \left(\frac{\sec r}{\sqrt{2 \sec^2 r - 2 \tan r \sec r - 1}} - 1 \right).$$

Hence we see that at the point $y_2 = b_1$, the derivative does not exist and $\partial B_v(x, r)$ is not smooth.

Remark 4.19. The proof of Theorem 4.13 gives more, namely the lines $l_1(y_2, r)$ and $l_2(y_2, r)$ are tangent to the $\partial B_v(x, r)$. To see this we first compute the slope of tangent lines to $B_v(x, r)$ at the points b_1, b_2 . We have by Remark 4.18

$$f'_2(b_1^+, r) = \cot r \left(\frac{\sec r}{\sqrt{2 \sec^2 r - 2 \tan r \sec r - 1}} - 1 \right).$$

Next, by (4.17), the slope of the line l_1 is $m_1 = \frac{\tan r}{-\sec^2 r + \tan r \sec r + 1}$. We claim that $m_1 = f'_2(b_1^+, r)$. To see this it is enough to show that

$$\frac{\tan^2 r + 1 - \sec^2 r + \sec r \tan r}{\sec r} = \frac{-\sec^2 r + \tan r \sec r + 1}{\sqrt{2 \sec^2 r - 2 \tan r \sec r - 1}}.$$

It is easy to check that for $0 \leq r \leq \pi/2$, both sides are equivalent to $\cos(\frac{r}{2}) \csc(\frac{\pi}{4} - \frac{r}{2}) \csc(\frac{\pi}{4} + \frac{r}{2}) \sin(\frac{r}{2})$.

Similarly, substituting b_2 from (4.16), gives

$$f'_2(b_2^+, r) = \cot r \left(\frac{\sec r}{\sqrt{2 \sec^2 r + 2 \tan r \sec r - 1}} - 1 \right).$$

Next, by (4.17), the slope of line l_2 is $m_2 = -\frac{\tan r}{\sec^2 r + \tan r \sec r - 1}$. We claim that $m_2 = f'_2(b_2^+, r)$. To see this it is enough to show that

$$\frac{-\tan^2 r - 1 + \sec^2 r + \sec r \tan r}{\sec r} = \frac{\sec^2 r + \tan r \sec r - 1}{\sqrt{2 \sec^2 r + 2 \tan r \sec r - 1}}.$$

It is easy to check that for $0 \leq r \leq \pi/2$, both sides are equivalent to $\cos(\frac{r}{2}) \csc(\frac{\pi}{4} - \frac{r}{2}) \csc(\frac{\pi}{4} + \frac{r}{2}) \sin(\frac{r}{2})$.

Theorem 4.20. For all $x \in \mathbb{H}^n$ and $r \in (0, \pi/2)$

$$B^n(x + (\sec^2 r - 1)x_n e_n, (\tan r)x_n) \subset B_v(x, r).$$

Proof. By symmetry, it suffices to consider the case $n = 2$.

Let us fix $x = i$. We claim that

$$B^2(i \sec^2 r, \tan r) \subset B_v(x, r).$$

By (4.15), the equation of $\partial B_v(i, r)$, is as follows

$$y_1 = \begin{cases} \cot r(1 + y_2 - 2\sqrt{y_2} \sec r) =: f_1(y_2, r), \\ \cot r(-1 - y_2 + 2\sqrt{y_2} \sec r) =: f_2(y_2, r). \end{cases}$$

Next we find the equation of $\partial B^2(i \sec^2 r, \tan r)$. Taking $|y - \sec^2 r| = \tan r$ gives

$$y_1 = \begin{cases} -\sqrt{-(y_2 - \sec^2 r)^2 + \tan^2 r} =: g_1(y_2, r), \\ \sqrt{-(y_2 - \sec^2 r)^2 + \tan^2 r} =: g_2(y_2, r). \end{cases}$$

By symmetry, it is sufficient to show that $f_2(y_2, r) \geq g_2(y_2, r)$. To see this we only need to show that $h(y_2, r) =: f_2(y_2, r)^2 - g_2(y_2, r)^2 \geq 0$. But $\frac{\partial^2 h(y_2, r)}{y_2^2} =$

$\frac{1}{y_2^{3/2}}(2y_2^{3/2} + \cos r - 3y_2 \cos r) \csc^2 r$. Denote by $h_2(y_2, r) = 2y_2^{3/2} + \cos r - 3y_2 \cos r$. But

$$\frac{\partial^2 h_2(y_2, r)}{y_2^2} = \frac{3}{2\sqrt{y_2}} > 0,$$

and $\frac{\partial h_2(\cos^2 r, r)}{y_2} = 0$. Moreover $h_2(\cos^2 r, r) > 0$ for $r \in (0, \pi/2)$. Therefore $\frac{\partial^2 h(y_2, r)}{y_2^2} \geq 0$. It is easy to check that $\frac{\partial h(\sec^2 r, r)}{y_2} = 0$, and hence $h(y_2, r) \geq 0$. \square

Theorem 4.21. *For all $x \in \mathbb{H}^n$, and $r \in (0, \pi/2)$*

$$(4.22) \quad B^n(x, x_n \sin r) \subset B_v(x, r).$$

Proof. Let $\lambda = \sin r$ and $y \in B^n(x, \lambda x_n)$. By domain monotonicity of the v -metric, we have for $B_x = B^n(x, x_n)$

$$v_{\mathbb{H}^n}(x, y) \leq v_{B_x}(x, y) \leq \arcsin \lambda,$$

where the last inequality follows by [KLWV, 3.3]. \square

Proof of Theorem 1.4. The proof follows from Theorems 4.13, 4.20 and 4.21. \square

Acknowledgements. The research of the first author was supported by UTUGS, The Graduate School of the University of Turku.

REFERENCES

- [AVV] G. D. ANDERSON, M. K. VAMANAMURTHY, AND M. VUORINEN, Conformal invariants, inequalities and quasiconformal maps. J. Wiley, 1997.
- [B1] A. F. BEARDON, The geometry of discrete groups. Graduate texts in math., Vol. 91, Springer-Verlag, New York, 1983.
- [B2] A. F. BEARDON: The Apollonian metric of a domain in \mathbb{R}^n . Quasiconformal mappings and analysis (Ann Arbor, MI, 1995), 91–108, Springer, New York, 1998.
- [CHKV] J. CHEN, P. HARIRI, R. KLÉN, AND M. VUORINEN: Lipschitz conditions, triangular ratio metric, and quasiconformal maps. Ann. Acad. Sci. Fenn. 40 (2015), 683–709, doi:10.5186/aasfm.2015.4039. arXiv:1403.6582 [math.CA].
- [GH] F. W. GEHRING AND K. HAG: The Ubiquitous Quasidisk, Mathematical Surveys and Monographs 184, Amer. Math. Soc., Providence, RI, 2012.
- [GP] F. W. GEHRING AND B. P. PALKA: Quasiconformally homogeneous domains. J. Analyse Math. 30 (1976), 172–199.
- [HIMPS] P. HÄSTÖ, Z. IBRAGIMOV, D. MINDA, S. PONNUSAMY AND S. K. SAHOO, Isometries of some hyperbolic-type path metrics, and the hyperbolic medial axis, In the tradition of Ahlfors-Bers, IV, Contemporary Math. 432 (2007), 63–74.
- [HMM] D. A. HERRON, W. MA, AND D. MINDA, Möbius invariant metrics bilipschitz equivalent to the hyperbolic metric. Conform. Geom. Dyn. 12 (2008), 67–96.
- [HKL] S. HOKUNI, R. KLÉN, Y. LI, AND M. VUORINEN, Balls in the triangular ratio metric. to appear in Proceedings of the international conference Complex Analysis and Dynamical Systems VI, Contemporary Math.
- [KL] L. KEEN AND N. LAKIC, Hyperbolic geometry from a local viewpoint. London Mathematical Society Student Texts, 68. Cambridge University Press, Cambridge, 2007. x+271 pp.
- [K1] R. KLÉN, Local convexity properties of quasihyperbolic balls in punctured space. J. Math. Anal. Appl. 342 (2008), 192–201. arXiv:0710.2973 [math.MG].
- [K2] R. KLÉN, Local convexity properties of j -metric balls. Ann. Acad. Sci. Fenn. Math. 33 (2008), no. 1, 281–293. arXiv:0710.2386 [math.MG].

- [K3] R. KLÉN, Close-to-convexity of quasihyperbolic and j -metric balls. *Ann. Acad. Sci. Fenn. Math.* 35 (2010), no. 2, 493–501. arXiv:1001.3923 [math.MG].
- [K4] R. KLÉN, Local convexity properties of balls in Apollonian and Seittenranta's metrics. *Conform. Geom. Dyn.* 17 (2013), 133–144. arXiv:1204.0329.
- [KLVW] R. KLÉN, H. LINDÉN, M. VUORINEN, AND G. WANG, The visual angle metric and Möbius transformations. *Comput. Methods Funct. Theory* 14 (2014), 577–608, arxiv.org/abs/1208.2871math.MG, DOI 10.1007/s40315-014-0075-x.
- [KMS] R. KLÉN, M. R. MOHAPATRA, AND S. K. SAHOO: Geometric properties of the Cassinian metric. arXiv:1511.01298 [math.MG], 16pp.
- [KRT] R. KLÉN, A. RASILA, J. TALPONEN, Quasihyperbolic geometry in Euclidean and Banach spaces. *J. Anal.* 18 (2010), 261–278.
- [KV1] R. KLÉN, M. VUORINEN, Inclusion relations of hyperbolic type metric balls. *Publ. Math. Debrecen* 81 (2012), no. 3-4, 289–311.
- [KV2] R. KLÉN, M. VUORINEN, Inclusion relations of hyperbolic type metric balls II. *Publ. Math. Debrecen* 83 (2013), no. 1-2, 21–42. arXiv:1105.1231 [math.MG].
- [S] P. SEITTENRANTA, Möbius-invariant metrics. *Math. Proc. Cambridge Philos. Soc.* 125 (1999), no. 3, 511–533.
- [Vu1] M. VUORINEN: Conformal invariants and quasiregular mappings. *J. Anal. Math.* 45 (1985), 69–115.
- [Vu2] M. VUORINEN, Conformal geometry and quasiregular mappings. Lecture notes in math. 1319, Springer-Verlag, Berlin, 1988.
- [Vu3] M. VUORINEN, Geometry of Metrics. *Proc. ICM2010 Satellite Conf. International Workshop on Harmonic and Quasiconformal Mappings (HMQ2010)*, eds. D. Minda, S. Ponnusamy, N. Shanmugalingam, *J. Analysis* 18 (2010), 399–424, arXiv:1101.4293 [math.CV].

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF TURKU, TURKU, FINLAND

E-mail address: parisa.hariri@utu.fi

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF TURKU, TURKU, FINLAND

E-mail address: riku.klen@utu.fi

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF TURKU, TURKU, FINLAND

E-mail address: vuorinen@utu.fi